Why Set Theory is Not an Acceptable Theoretical Foundation for Mathematics

Currently, Set Theory is the theoretical foundation of mathematics. As the standard theoretical foundation, Set Theory supplies vocabulary and axioms for proofs that justify fundamental math theorems. In other words, explanations of *why* numbers, arithmetic and other features of math work the way they do must be made in the language of Set Theory, and justified using axioms and theorems of Set Theory.

> "Many different axiomatic theories have been proposed to serve as a foundation for set theory, but, no matter how they differ at the fringes, they all have as a common core the fundamental theorems which mathematicians need in their daily work." —*Mendelson*[1]

The definitions and axioms of the theoretical foundation have a critical impact on mathematical knowledge since the proof of *any result* depends on them. The results that have been proven according to one foundation may not be true or proven according to a different foundation. Thus, there is strong incentive to support the reigning foundation. The work to reevaluate the accomplishments of the past, the sunk costs of the intellectual investment already made, and uncertainty about the virtues of the new foundation all provide a strong resistance to change.

Set Theory, however, is an unacceptable foundation for mathematics. An evaluation and improvement is critical for the future of mathematics. Set Theory enforces a bias that inhibits discovery. Also, its definitions and axioms have a tortured relationship to the numbers and functions that it is supposed to explain which alienates potential students. However, the main reason why I am daring to challenge the *status quo* is that Set Theory fails to supply the intellectual integrity and justice that is worthy of mathematics.

Support of this claims follows. The organization of the development is: i) assumptions and definitions of this paper, ii) specific claims with corresponding justification, iii) assessment, and iv) conclusion. [2]

Assumptions and Definitions

The assumptions and some key definitions for the following analysis are as follows.

1.) Written English language can be used: a) as a means to communicate and b) as a known system of symbols and meanings.

Every academic paper published in English assumes 1.a. Why would anyone do the work of preparing a paper and publishing it if he or she did not expect it to communicate something.

1.b is true in cases where the following expectation is true. Scholars who have achieved an understanding of Set Theory also have training in grammar and have experience with attempts to express meanings through the symbols of language.

An Accepted Tenet of Language

> The meaning of a word is established with respect to a context.

The relationship of a meaning to its context is demonstrated by noticing that the word "set" has one meaning in Set Theory, another meaning in tennis, and another meaning in bricklaying. Also, "work" has a specific meaning in the context of physics that is different from its meaning in the context of commerce, and "point" has a specific meaning in the context of geometry that is different from its meaning in the context of debating. If the relationship between a word and its meaning was always the same regardless of context, then this could not be the case.

Claim 1 and Justification

> A theory establishes a context.

This follows from the definition of context. From the American Heritage Dictionary[3]: "**context** (n.) 1. The setting of words and ideas in which a particular word or statement appears."

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[1] *Ref. 1, as specified on page 2, p. 173.*

[2] *The 2000 Mathematics Subject Classification for the primary and secondary subjects of this paper are respectively: 03-XX Mathematical logic and foundations: 1) 03A05 Philosophy and critical, and 2) 03EXX Set Theory.*

[3] The American Heritage Dictionary. *Based on the new second college edition. 1983. Houghton Mifflin Company, Boston, MA.*

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2.) The language of informal Set Theory is developed within the broader context of general proper (American) English. Thus, if meanings and punctuation are not clearly defined in informal Set Theory, then the default, governing authority is English.

3.) A character string effects communication if and only if it refers to a meaning.

4.) ***Definition:*** *Name* ≡ a character string and the meaning it refers to.

5.) ***Definition of a language practice:*** Single quotes offset a name when referring to the unique identity of its character string. Double quotes offset a name when referring to the unique identity of the name itself, both its symbol and meaning. Otherwise, names refer to their associated meaning.

6.) The meaning of "=" ["to equal"] in English is: A = B if and only if both 'A' and 'B' refer to the same meaning.

7.) Readers are familiar with the first-order Predicate Calculus and Set Theory.

8.) The following references are reliable authorities for what constitutes Set Theory:

Reference 1: a college-level text with a rigorous, formal development
Mendelson, Elliott. 1979. *Introduction to Mathematical Logic, 2nd edition*. D. Van Nostrand Company, New York.

Reference 2*:* a publicly available reference (from Borders Bookstore) that provides a more accessible development.
Stoll, Robert. 1963. *Set Theory and Logic.* Dover Publications, Inc., New York.

One of the difficulties with Set Theory is the lack of a single standard.

Stoll (1963) distinguishes between intuitive Set Theory—as originated by G. Cantor (1845-1918), informal axiomatic Set Theory as it was developed by Zermelo (1908), Fraenkel and Skolem [called ZF Set Theory] and formal axiomatic Set Theory. Mendelson (1979) introduces set-theoretic concepts and relationships in his introduction as "basic ideas and results used in the text." This development parallel's Stoll's intuitive approach. In a later section, Mendelson defines formal Set Theory as proposed by von Neumann (1925, 1928) and revised by Robinson (1937), Bernays (1937-1954), and Gödel (1940) [called NBG Set Theory].

For the purposes of this paper, let '*informal Set Theory'* refer to Stoll's intuitive version of Set Theory which is equivalent to Mendelson's basic ideas and results. Let '*formal Set Theory'* refer to NBG Set Theory as set forth in Chapter 4 of Ref. 1. Scholars generally agree that the ZF and NBG versions of Set Theory lead to equivalent results.

Informal Set Theory has known problems associated with some well-publicized paradoxes. Formal Set Theory was established to prohibit the paradoxes and obtain more rigor.[4] However, informal Set Theory has not been abandoned.

Most mathematicians do not have training or formal study in formal Set Theory, and yet they use the concept of a set. Given the sidelined, non-essential role of formal Set Theory in graduate mathematics, common references to sets usually invoke informal Set Theory. Also, ongoing use of informal Set Theory is evidenced by the fact that both Stoll and Mendelson find it necessary to set forth the basics of informal Set Theory in their respective texts.

Therefore, the current status is that both informal and formal Set Theory are employed as a theoretical basis for work in mathematics.

9.) If a theoretical foundation is acceptable for a given discipline, then: a) its assumptions are explicit, non-contradictory, and well defined; and b) it assists with understanding and the development of knowledge for the society as a whole.

Claim 2 and Justification

A theory must avoid producing ambiguity and/or contradiction.

Meanings should be clear and consistent. For example, the meaning of "black" and "white" should not be different in some cases and the same in some cases. The rationale for accepting the standard put forward by this claim follows.

[4] *Ref. 1, p. 173 and Ref. 2, p. 289.*

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Ambiguity allows different people to reach different conclusions [by definition of "ambiguous"]. And contradiction allows the formation of statements A and ¬A such that both are true. This allows any statement in the theory to be proved through the tautology $A \rightarrow (\neg A \rightarrow B)$. Neither of these situations assists with understanding and the development of knowledge for society as a whole. Thus, if Set Theory produces ambiguity and/or contradiction, then it is not an acceptable foundation for mathematics by Assumption 9 [Ass 9].

Consequently, it will be sufficient in the following development of problematic issues to show that aspects of Set Theory result in ambiguity and/or contradiction.

Defects of Set Theory

The particular problems with Set Theory to be discussed in this paper are listed below.

Formal Set Theory

I. Definition by circular reference
II. Founding mathematics on nothing

Informal Set Theory

III. Violation of the difference between part and whole
IV. Dependence on an undefined operation

Although Set Theory has supported a lot of work to date, it is unsound as the primary theoretical setting for the number-related part of mathematics. Problems with informal Set Theory go beyond the paradoxes. And formal Set Theory has its own fundamental flaws. The defects of informal and formal Set Theory determine that Set Theory is not an acceptable foundation for mathematics.

Whether or not Set Theory can be repaired will not be addressed; rather, the issues raised in this paper support a call to abandon Set Theory and find a different theory for the foundation of mathematics.

Formal Set Theory

Definition by circular reference

The following argument will focus on definitions to show that formal Set Theory defines numbers via circular reference.

On the basis of Assumption 7, the reader is aware of the how formal Set Theory is used to define natural numbers. [Stoll provides a presentation on p299; Mendelson mentions equivalence classes and cardinal numbers on p196]

In the beginning of Chapter 4, Axiomatic Set Theory, in Reference 1, the definition of formal Set Theory begins with the following words (emphasis added).

> 1. An Axiom System
> …We shall describe a **first-order theory** NBG. ... NBG has a single predicate letter A_2^2, but no function letters or individual constants. ... Let us define equality in the following way. ... We shall now present the proper axioms of NBG, interspersing among the statement of the axioms some additional definitions and various consequences of the axioms.[5] —*Mendelson*

What we see is "NBG" being introduced as a new term. All words come from English, a general context, with the following exceptions: "first-order theory," "predicate letter," "A_2^2," "function letters," "constants," and "proper axioms." In Reference 1, these terms get their respective meanings from the theoretical context given by Chapter 2, Quantification Theory, where the notion of "first-order theory" is introduced.

Is the meaning of "first-order theory" critical to the definition of formal Set Theory? Yes. This is supported in the following quote by Stoll to introduce first-order theories:

> In this chapter we give an introductory account of modern investigations pertaining to formal axiomatic theories—that is, axiomatic theories in which there is explicitly incorporated a system of logic.[6]

Defining NBG as a first-order theory determines that it includes an explicit system of logic, with accompanying axioms and theorems.

To get a better understanding of "first-order theory" and the other logic-related terms used to define NBG, consider the following section in Ref. 1, Chapter 2. Emphasis is added: boldness emphasizes

[5] *Ref. 1, p. 173*
[6] *Ref. 2, p. 373*

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number-related terms, and italics emphasize set-theoretic terms.

> 3. First-Order Theories
>
> …The symbols of a first-order theory K are essentially those introduced earlier in this chapter: the propositional connectives [¬, →]; the punctuation marks (,), , (the comma is not strictly necessary but is convenient for ease in reading formulas); ***denumerably many*** individual variables $\mathbf{x_1}$, $\mathbf{x_2}$,…; a finite or ***denumerable*** *non-empty set* of predicate letters $\mathbf{A_j^n}$ (**n, j≥1**); a finite or ***denumerable,*** possibly *empty, set* of function letters $\boldsymbol{f_j^n}$ (**n, j≥1**); and a finite or ***denumerable,*** possibly *empty, set* of individual constants $\mathbf{a_i}$ (**i≥1**). … Different theories may differ in which of these symbols they possess.
>
> The definitions given in Section 1 for term, wf [well-formed formula], and for the propositional connectives [and, or, ↔], are adopted for any first-order theory.
>
> The axioms of K are divided into **two** *classes*: the logical axioms and the proper (or non-logical) axioms.
>
> Logical axioms: ...
>
> Proper axioms: These cannot be specified, since they vary from theory to theory. A first-order theory in which there are no proper axioms is called a first-order predicate calculus.
>
> The rules of inference of any first-order theory are (i) Modus ponens: ... [and] (ii) Generalization: ...[7] —*Mendelson*

The new terms introduced here are "first order theory" and "K." We assume that the set-theoretic terms gain their meaning from the context of informal Set Theory; otherwise, it would be a very obvious case of circularity. Someone may still be able to make the case for circularity based on these set-theoretic terms, however, the argument here depends on the number-related terms. Look more closely at these number-related terms.

The definition of "denumerable" is given on page 8 of Reference 1: "A set X is *denumerable* if it is equinumerous with the set of positive integers." Thus, the meaning of "positive integers" is crucial to the meaning of "denumerable." The other number-related terms have issues explained as follows.

The atomic words defining a first-order theory are a_i (constant letters), x_i (variables), f_j^n (function letters), and A_j^n (predicate letters) where i, j, and n refer to counting numbers. However, technically, according to the founders these words do not have any meaning, they are just symbols. Meaning is assigned through an interpretation. This is best said by Stoll in the following quote:

> 1. Formal Axiomatic Theories. In order to achieve precision in the presentation of a mathematical theory, symbols are used extensively. A formal theory carries symbolization to the ultimate in that all words are suppressed in favor of symbols. Moreover, in a formal theory the symbols are taken to be merely marks which are manipulated according to given rules which depend only on the form of the expressions composed from the symbols.

So the "words" of any formal theory do not represent meaning; they are just marks on a page. At least this is the plan—the meaning for these words will vary from interpretation to interpretation for any given theory. However, in practice, the symbols are not just marks devoid of meaning because the symbols for numbers do indeed refer to number meanings.

In the context of Quantification Theory, the system for creating symbols involves a mapping of numbers onto copies of a given letter: 'a', 'x', '*f*', or 'A.' Evidence that these numbers are not just different unique marks is given by the fact that they are used to:

a) Impose an ordering of previous, concurrent, and subsequent in lists of these words
b) Supply the means to do proof by induction
c) Establish correspondence between the symbols and the meanings of a given interpretation
d) Implement Gödel's system of arithmetization

Thus, in the context of Quantification Theory, the names of numbers—both symbol and meaning—play a significant role.

Let's double-check whether the definition of "first-order theory" relies on number meanings by looking at the definition of first-order theory given by Stoll [p395]. Emphasis is added as specified previously.

[7] *Ref. 1, p. 58-59.*

We now turn to a precise description of a first-order theory K. The primitive symbols are the following.

(I_s) An infinite sequence of individual variables, $\mathbf{a_0}$, $\mathbf{a_1}$, $\mathbf{a_2}$, …

(II_s) A *set* of logical constants consisting of

(a) The logical symbols of the predicate calculus, parentheses, and a comma,

(b) The equality symbol "=."

(III_s) A *set* of **mathematical** constants consisting of

(a) A *set* of individual constants,

(b) For each **positive integer n**, a *set* of **n-place** predicate (or relation) symbols

(c) For each **positive integer n**, a *set* of **n-place** operation symbols.

… —*Stoll*

As you can see, numbers are used to define a first-order theory here also.

The founders believed that the vocabulary of natural language included counting numbers; however, one of the ultimate goals was to better understand numbers and numeric relationships. Thus, Quantification Theory provides a general context in which to develop a formal theory of mathematics. Both formal Number Theory and Set Theory were constructed to address this need, however they defined counting numbers rather than using them from natural language. Consider that if numbers are primitives taken from English for Quantification Theory then they are available as primitives in any sub-context of Quantification Theory, including each formal first-order theory, especially Set Theory.

The result currently is this. Numbers are concepts that gain their meaning from the context of mathematics. The number-part of mathematics is formally defined through formal Set Theory. Set Theory is defined using "first-order theory". And "first-order theory" is defined using numbers which are defined by mathematics, which is defined by formal Set Theory, which is defined using "first-order theory," which is defined using numbers, which are defined by mathematics which is defined...

Therefore, formal Set Theory does not define the number part of mathematics properly. It uses circular reference. And definition through circular reference does not work. It is as useful as saying "I define a googlethorpe to be a googlethorpe."

This means that with formal Set Theory as the foundation of mathematics, the number part of mathematics is not well defined. Consequently, it has ambiguity. By Claim 2, formal Set Theory is not acceptable as a theoretical foundation for mathematics.

Founding mathematics on nothing

A remaining problem that disables the effectiveness of formal Set Theory is that sets are not constituted of anything substantive. Sets are determined by operations applied to the null set, a single atomic concept that is a form of nothing.

The null set is defined in formal Set Theory as the set that has no members, or rather the set which satisfies: $(\exists x)(\forall y)\neg(y\in x)$.

Does the null set really cause any problems for formal Set Theory? After all, so many mathematicians have used it without doubting the validity of what they were doing. A misunderstanding exists, however, about the difference between the somethingness of the concept-of-nothing outside the context of Set Theory and the nothingness of the underlying meaning within the context of Set Theory. This misunderstanding has been propagated through convention and standard practice.

Consider the following.

The objects of formal Set Theory are:

a) The null set, ø

b) Results of applying operations to (a)

c) Results of applying further iterations of operations to (a) and (b) where operations include: set function (pairing), intersection, complement, n-tuple generation, sum set generation, power set generation, replacement, successor set generation, etc.

Is this manipulation of what is essentially nothing a productive endeavor? For example, in the context of a database, any operation on a null value such as addition, subtraction, concatenation, or parsing returns a null value. In formal Set Theory, however, set creation operations performed on null values return non-null values.

So far, I have been using the liberty granted by Stoll to analyze the object language of formal Set

Theory with English as my metalanguage.[8] According to mathematical logic, however, well-formed formulas of a formal theory have meaning only when an interpretation is given for the symbols.[9] Thus, consider the following interpretation.

The domain is a single blank list from the context of the English language such that it is indicated in a character string by two blank spaces ' '. The use of blank spaces to indicate a form of nothing in language creates problems for our perception, so the presence of these spaces will be accentuated by underlining, '__'. Note that a unique list in English is determined by the items on the list, which is similar to a general tenet of Set Theory that a set be determined by its members. Also, lists of lists occur in English just like sets of sets occur in Set Theory.

Assign to '∈' the meaning "is an item on the following list."

Since "∈" is the only predicate letter of formal Set Theory and formal Set Theory does not use any constant letters or function letters, this completes the definition of an interpretation for formal Set Theory.

Now consider whether or not "x ∈ x":

__ is an item on the following list __

Imagine a list such that each entry on the list is "__". For instance, with the Pairing Axiom, we might want to make a list: __, __. But, this is just a blank list. So any such list is equivalent to the blank list "__". Since "__ is an item on the following list __, __, __" is true [for this style of list, items are separated by a comma followed by a space; and the blank list is between these separators] and __, __, __ = __, then "__ is an item on the following list __" is true [by the Replacement Theorem of Math Logic, Mendelson Corollary 2.21].

Then again, "__" is a blank list, meaning that "__" does not include any items, not even itself. So "__ is not an item on the following list __" is also true.

Consequently, since a statement and its negation are both true, so is every other statement of the interpretation, including the Set Theory axioms. And thus, the interpretation is a model.

Possibly, this model of formal Set Theory causes problems that should be investigated.[10] The main point here, however, is that the null set in the context of Set Theory is equivalent to a blank list in the context of language. What value does a communication piece have that uses blank lists as the subjects and objects of most sentences?

Numbers are well-defined, substantive noun meanings, yet formal Set Theory defines them using the null set as the only primitive noun meaning. The null set is defined by the absence of anything in the context of Set Theory. And "absence of anything" is equivalent to "nothing."

Manipulating nothing in various ways does not assist with developing knowledge of mathematics for the society as a whole. Therefore, formal Set Theory is not an acceptable theoretical foundation for mathematics [Assumption 9].

Informal Set Theory

How sets are defined and constructed

The quotes below capture the permitted rules for defining and constructing sets according to informal Set Theory. They provide a reference for the following arguments made with respect to informal Set Theory.

> "That uniquely determined set whose members are the objects $x_1, x_2, \ldots, x_n$ will be written $\{x_1, x_2, \ldots, x_n\}$. In particular, $\{x\}$, a so-called *unit set*, is the set whose sole member is x."[11]

> "*The intuitive principle of abstraction.* A formula P(x) defines a set A by the convention that the members of A are exactly those objects as such that P(a) is a true statement. Because sets having the same members are equal, a given formula determines exactly one set which, in mathematics, is usually denoted by $\{x|P(x)\}$, read 'the set of all x such that P(x).'"[12] —*Stoll*

[8] *Ref. 2, p. 402.*

[9] *Ref. 1, p. 50.*

[10] *Consider implications of Ref. 1, Corollary 2.15. Also consider that this model shows that the null set can be both nothing and everything in the context of Set Theory, which is a form of ambiguity.*

[11] *Ref. 2, p5. Confirmed in Ref. 1, p5: "A* set *is a collection of objects. The objects in the collection are called* elements *or* members *of the set, and we shall write "x∈y" for the statement that x is a member of y. … Given any objects $b_1, \ldots, b_k$, the set which contains $b_1, \ldots, b_k$ as its only members is denoted $\{b_1, \ldots, b_k\}$.*

[12] *Ref. 2, p. 6-7. Confirmed in Ref. 1, p. 6 as a special case of an* n-place relation.

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Violation of the difference between part and whole

A defect of informal Set Theory is the freedom it allows to define a set that properly contains itself, i.e., a set that contains other sets *and* itself. By definition, a proper part does not equal the whole. Yet, Set Theory produces the result that a proper part is equal to the whole. Consider the following reasoning with respect to informal Set Theory.

> Define a set Cs such that Cs contains all sets whose respective names begin with the letter 'C'. [This is possible through the property method of set definition.] Thus, Cats ∈ Cs, and Cars ∈ Cs [by definition of "Cs"]. Now, "Cs" is the name of a set and its name begins with 'C'. Therefore, Cs satisfies the standard given by its own definition, so that Cs ∈ Cs.
>
> In the following, let '…' represent all sets whose respective names begin with 'C' other than Cats, Cars, and Cs. Consider, then, that {Cats, Cars,…, Cs} is a set by the list method of set definition. For any set Y, Y ∈ Cs if and only if Y ∈ {Cats, Cars,…, Cs} [by the respective definitions of "Cs" and "{Cats, Cars,…, Cs}"]. By definition of "=" in Set Theory then:
>
> Cs = {Cats, Cars,…, Cs}[13]
>
> Note that "Cs" is the whole name on one side of '=' and only part of the name on the other side. Also observe that by the definition of = [Ass 4], the character strings on both sides of "=" refer to the same meaning. The principles of Set Theory establish the conditions in which two set meanings are the same ('=' is used to denote this relation) they do not redefine the meaning of "=". Thus, according to Set Theory, the meaning of "{Cats, Cars,…, Cs}" is the same as the meaning of "Cs."
>
> Now, "{Cats, Cars,…, Cs}" is built from the following meaning components: Cats, Cars, …, and Cs. Thus, the whole meaning of "{Cats, Cars,…, Cs}" includes Cs as a part as well as Cats, Cars, and …, as parts. These parts are put together in an additive way with punctuation to form the whole meaning of "{Cats, Cars,…, Cs}". In "Cs = {Cats, Cars,…, Cs}," "Cs" refers to the whole meaning on one side of "=" and a proper part of the meaning on the other side. By making "Cs = {Cats, Cars,…, Cs}" true, informal Set Theory violates the distinction between part and whole.

[13] *Ref. 1, p. 173: Definition of "=" in formal Set Theory: $x=y \leftrightarrow (\forall z)\,(z\in x \leftrightarrow z\in y))$. Ref. 2, Definition of "to equal" in informal Set Theory: "sets having the same members are equal."*

A set that properly contains itself violates the distinction between part and whole. By definition, a proper part does not equal the whole. Yet, Set Theory produces the result that a proper part is equal to the whole. Thus, informal Set Theory produces contradiction on a fundamental level.

This problem is possible because there are no restrictions regarding context. A property can be drawn from and applied to any combination of contexts. Names are valid objects as well as the meanings they refer to.

This problematic flaw allows the definition of the well-known Set Theory paradoxes, while being the source of more widespread trouble as well. Therefore, by Claim 2, Set Theory is not acceptable as a theoretical foundation for mathematics.

Dependence on an undefined operation

What determines a set? Is a set completely determined by its members? Experts say yes, but in actuality, the answer is no. A close examination of construction practices in informal Set Theory reveals that a set is determined by braces, '{' and '},' as well as constituent members. Consider the following.

> Define a set called u that contains only the members: 1, 2, and 3. [This is possible through the list method of set definition.]. If the identity of a set is completely determined by its members, then 'u' and '1, 2, 3' refer to the same meaning in the context of Set Theory. ["u" is the name; "1, 2, 3" is the standard for the meaning of "u." That 'u' refers to "1, 2, 3" as a group and not individually, as is the case in say Algebra, is given by the context, Set Theory.]
>
> If 'u' and '1, 2, 3' both refer to the same meaning, then u = 1, 2, 3 by the definition of = in English [Ass 6]. Also, u = 1, 2, 3 by the definition of = in Set Theory since for any set Y, Y ∈ u if and only if Y ∈ 1, 2, 3. In other words, no elements other than 1, 2, and 3 are elements of the set indicated by 'u' or the set indicated by '1, 2, 3'.

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In English, the symbols '{' and '}', called braces, do not denote meaning; they are punctuation marks. Using braces around '1, 2, 3' helps clarify that the character string is referring to 1, 2, 3 as a group, but the meaning of '1, 2, 3' is the same as the meaning of '{1, 2, 3}' in English. If braces are supposed to have meaning not specified by English, then a definition should be supplied as a part of Set Theory [Ass 2], but no such definition has been supplied. The list method of effecting or building a set specifies the use of braces as a syntax rule.

So according to rules in the larger context, English, 1, 2, 3 = {1, 2, 3} by the definition of = [Ass 6]. Also, 1, 2, 3 = {1, 2, 3} by the definition of = in Set Theory since for any set Y, Y ∈ 1, 2, 3 if and only if Y ∈ {1, 2, 3}. No elements other than 1, 2, and 3 are members of the set indicated by '1, 2, 3' or the set indicated by '{1, 2, 3}'. Therefore, since u = 1, 2, 3 and 1, 2, 3 = {1, 2, 3}, u = {u} [this property of = follows from definition, Ass 6].

However, according to informal Set Theory, u has a different meaning than {u}. Not u = {u}, rather u ∈ {u}. Based on the language practices of Set Theory, meaning is associated with the punctuation marks '{' and '}', and the meaning of these marks also has a role in determining a set.

In the context of Set Theory, braces denote a function.[14] This function, which we'll call the set function, acts on multiple individual elements and produces a single unit called a set. Just as 'f(x, y)' denotes the result effected by applying function f to x and y, '{x, y}' denotes the result effected by applying the set function to x and y.

Revealing the set function and its role in Set Theory leads to the following issues:

1. Either the set function has an arbitrary number of arguments, or Set Theory has infinitely many set functions, one for each n-many arguments, e.g. $\{x_1, x_2, \ldots, x_n\}$ or $\{x_1\}$, $\{x_1, x_2\}$, ... The situation is ambiguous.

2. The set function can be applied to nothing, null values, to produce something. In other words, f( ), or rather {} in this case, is an object of the theory. This contradicts what is acceptable according to intuitive, informal mathematics.

Legally, Set Theory (both formal and informal versions) avoids the adoption of a beginning, simple function to employ in effecting sets. The notion of a function is actually a type of set itself and it is defined later in the development. In practice, Set Theory depends on the set function.

The reason why Set Theory seems to be so useful as a theoretical foundation is because the acts of aggregating, separating and comparing are so basic to conceptual analysis. Set Theory, however, relies on an undefined construction process to develop the primary objects in its domain. This lack of definition and lack of clarity is a form of ambiguity. Thus, by Claim 2 Set Theory is not an acceptable foundation for mathematics.

Assessment

The violation of the distinction between part and whole is a system failure. The system integrity of Set Theory mathematics is further undermined by the unrestricted use of undefined operations. And, the substantive value of the system is nullified by its dependence on circular reference and the null set.

What practical value does this flawed system have? Does Set Theory support research and pedagogy in the field of mathematics so that both are productive? In other words, does Set Theory provide the primitive concepts, capabilities, axioms and definitions that working mathematicians use to further math knowledge, and do teachers use them to educate students about numbers?

With respect to research, work in mathematics addresses Set Theory for key definitions and the justification of key theorems, but the majority of work done in calculus, algebra, and analysis is done outside the context of Set Theory. This is supported by the following quotes:

> "Formalized mathematics, to which most philosophizing has been devoted in recent years, is in fact hardly to be found anywhere on earth or

[14] *Confirmed by the following introduction of braces to denote a function in formal Set Theory: "Since we have the uniqueness condition for the unordered pair, we can introduce a new function letter g(x, y) to designate the unordered pair of x and y. We shall write {x, y} for g(x, y)." —Ref. 1, p. 175.*

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> in heaven outside the texts and journals of symbolic logic."—*The Mathematical Experience*[15]

> "The loss of truth, the constantly increasing complexity of mathematics and science, and the uncertainty about which approach to mathematics is secure have caused most mathematicians to abandon science. With a 'plague on all your houses' they have retreated to specialties in areas of mathematics where the methods of proof seem to be safe." —*Morris Kline*[16]

The following example shows how work goes on in spite of Set Theory. In a college calculus textbook, the following set-theoretic definition of a function is given:

> "Definition: A function is a set of ordered pairs of numbers (x, y) with the following property: to each value of the first variable (x) there corresponds a unique value of the second variable (y)."

After 3 paragraphs comes the following assertion.

> "The next example shows how to describe functions without set-builder notation. Every function is determined by two things: (1) *the domain* of the first variable x and (2) *the rule* or condition that the pairs (x, y) must satisfy to belong to the function. We can therefore describe a function completely by giving its domain and rule." —Thomas and Finney (1979)[17]

All work in calculus is conducted using the second approach, without set-theoretic reasoning.

With respect to pedagogy, an attempt was made to use Set Theory as a basis for teaching mathematics in primary and secondary schools under the label of New Math. The general consensus is that it was not productive.

Any theory that is a foundation of mathematics becomes employed in providing explanations and proof. Schoolteachers who need to provide reasons and explanations to their students are unable to do so because the explanations (or proofs) are not accessible to them or their students. They are put in the position of forcing their students to accept what they say on the basis of authority alone. They have no means to overcome whatever natural resistance to understanding and learning that is present in a student's mind.

A current education goal of society is that students who graduate from high school should have a basic level of math proficiency. The decision to let Set Theory govern the foundation of mathematics is preventing successful movement towards the realization of this goal.

Set Theory is not sound as a theoretical foundation and it has little practical value for research and pedagogy in mathematics.

Conclusion

Each of the defects noted above:

- Definition by circular reference
- Founding mathematics on nothing
- Violation of the distinction between part and whole
- Use of undefined operations
- Lack of support for research and pedagogy

presents an independent reason why Set Theory is not an acceptable theoretical foundation for mathematics. The lack of clarity and understanding that accompany this approach are unacceptable for the theoretical foundation of mathematics. An effort should be made to consider other candidates for this role.

In my paper entitled *Understanding Numbers*, I submit such a candidate for consideration. The axiom of this development is "ones and the ability to add are present in the context of mathematics." All known numbers, as well as new numbers, are constructed and defined. (General principles for construction and definition are addressed in my paper *Principles for Working Together on Knowledge*.)

Whether the foundation proposed in *Understanding Numbers* is acceptable or not, mathematics should be rebuilt on a foundation that is more reliable. Currently, mathematics is like the Tower of Pisa. Too much energy and attention must be spent to preserve the desired relationship of the structure to its foundation. A better foundation would provide a greater capability for development. It would also provide clarity, rather than ambiguity, for a greater number of students.

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[15] The Mathematical Experience *by Philip J. Davis and Reuben Hersh. 1981. p. 347-348.*

[16] Mathematics, The Loss of Certainty *by Morris Kline, Oxford Press: Oxford. 1980. p. 7.*

[17] *George B. Tomas, Jr. and Ross L. Finney. 1979.* Calculus and Analytic Geometry, 5th ed. *Addison-Wesley Publishing Co., Reading, MA.*

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